CS 4649/7649
Robot Intelligence: Planning

Kalman Filtering

Sungmoon Joo

School of Interactive Computing
College of Computing
Georgia Institute of Technology

*Slides based in part on Dr. Mike Stilman's slides

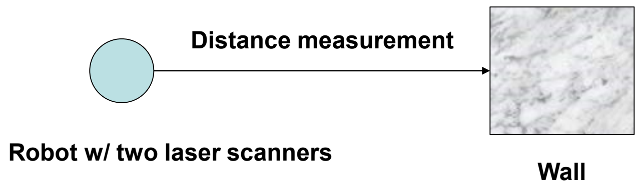
Kalman Filter

- R. Kalman born in Hungary in 1930
  - BS & MS from MIT, ScD from Columbia
  - Devised filter in 1960
  - Changed control theory forever
  - Awarded National Medal of Sciences Oct. 7 2009

Kalman Basics: Estimating Distributions

Model #1: Robot is not moving

Two measurements with different sensors… What is the best estimate of the distance?

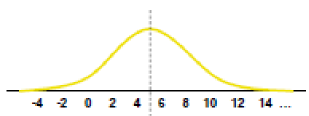


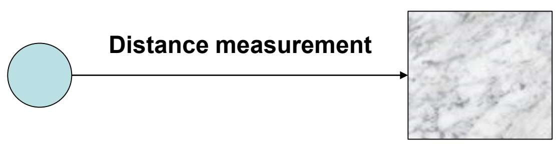
Kalman Basics: Estimating Distributions

a measurement with sensor#1

z_1, σ_{z1}^2

$$\hat{x}_1 = z_1$$
$$\hat{\sigma}_1^2 = \sigma_{z1}^2$$





Kalman Basics: Estimating Distributions

a measurement

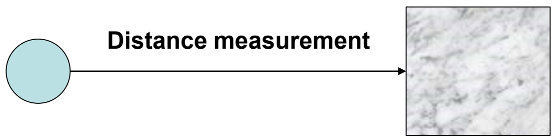
$$z_1, \sigma_{z1}^2$$

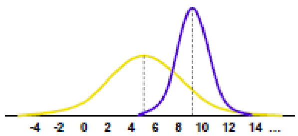
$$\hat{x}_1 = z_1$$
$$\hat{\sigma}_1^2 = \sigma_{z1}^2$$

another measurement with sensor #2

$$z_2, \sigma_{z2}^2$$

$$\hat{x}_2 = ?$$
$$\hat{\sigma}_2^2 = ?$$



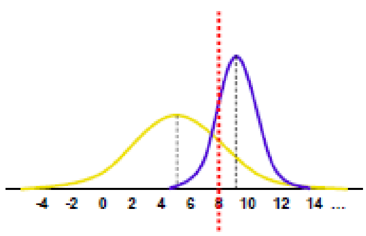


Conditional Density: Combine Estimates

$$\hat{x}_2 = [\frac{\sigma_{z2}^2}{\sigma_{z1}^2 + \sigma_{z2}^2}]z_1 + [\frac{\sigma_{z1}^2}{\sigma_{z1}^2 + \sigma_{z2}^2}]z_2$$

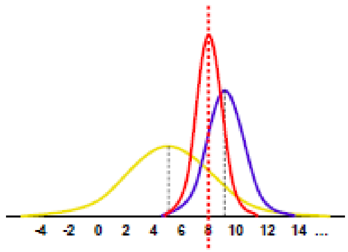
$$\hat{x}_2 = \hat{x}_1 + K_2(z_2 - \hat{x}_1)$$

$$K_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_{z2}^2} \quad , \sigma_1 = \sigma_{z1}$$

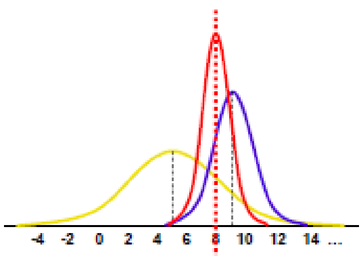


Conditional Density: Combine Variances

$$\frac{1}{\sigma_2^2} = \frac{1}{\sigma_{z1}^2} + \frac{1}{\sigma_{z2}^2} \quad \text{or} \quad \sigma_2^2 = \frac{\sigma_{z1}^2 \sigma_{z2}^2}{\sigma_{z1}^2 + \sigma_{z2}^2}$$



Kalman Basics



a measurement

z_1, σ_{z1}^2

$$\hat{x}_1 = z_1$$
$$\hat{\sigma}_1^2 = \sigma_{z1}^2$$

another measurement

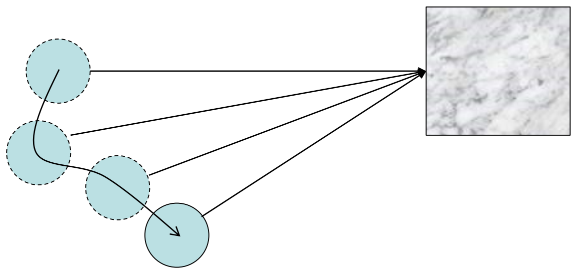
z_2, σ_{z2}^2

$$\hat{x}_2 = \hat{x}_1 + K_2(z_2 - \hat{x}_1)$$
$$K_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_{z2}^2}$$
$$\hat{\sigma}_2^2 = \frac{\sigma_{z1}^2 \sigma_{z2}^2}{\sigma_{z1}^2 + \sigma_{z2}^2}$$

What if the robot is moving?

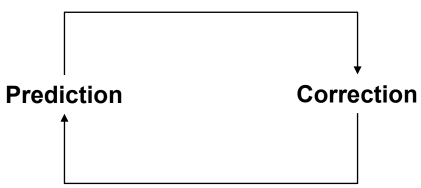
Model #2: Robot is moving

- **How do we incorporate the motion in our estimation?**
- **Kalman Filter can include a LINEAR model of motion**



Kalman Filter Concept

Predict(using ‘model’) the new state and its uncertainty



Correct prediction with new measurement

Simple Process Model

$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k-1} \\ x_{2,k-1} \end{bmatrix} + \begin{bmatrix} w_{1,k-1} \\ w_{2,k-1} \end{bmatrix}$$

$$\mathbf{x}_k = \mathbf{A}\mathbf{x}_{k-1} + \mathbf{w}_{k-1}$$

Measurement Model

$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k-1} \\ x_{2,k-1} \end{bmatrix} + \begin{bmatrix} w_{1,k-1} \\ w_{2,k-1} \end{bmatrix}$$

$$\mathbf{x}_k = \mathbf{A}\mathbf{x}_{k-1} + \mathbf{w}_{k-1}$$

$$\begin{bmatrix} z_{1,k} \\ z_{2,k} \end{bmatrix} = \begin{bmatrix} H_{1,1} & 0 \\ 0 & H_{2,2} \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} v_{1,k} \\ v_{2,k} \end{bmatrix}$$

$$\mathbf{z}_k = \mathbf{H}\mathbf{x}_k + \mathbf{v}_k$$

Measurement Model

$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k-1} \\ x_{2,k-1} \end{bmatrix} + \begin{bmatrix} w_{1,k-1} \\ w_{2,k-1} \end{bmatrix}$$

$$\mathbf{x}_k = \mathbf{A}\mathbf{x}_{k-1} + \mathbf{w}_{k-1}$$

$$\mathbf{Q} = E(\mathbf{w}\mathbf{w})^{\mathrm{T}} = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}$$

$$\mathbf{R} = E(\mathbf{v}\mathbf{v})^{\mathrm{T}} = \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix}$$

$$\begin{bmatrix} z_{1,k} \\ z_{2,k} \end{bmatrix} = \begin{bmatrix} H_{1,1} & 0 \\ 0 & H_{2,2} \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} v_{1,k} \\ v_{2,k} \end{bmatrix}$$

$$\mathbf{z}_k = \mathbf{H}\mathbf{x}_k + \mathbf{v}_k$$

Kalman Filter

Initialize

$$\hat{\mathbf{x}}_0 \quad \mathbf{P}_0 = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}$$

Prediction

$$\hat{\mathbf{x}}_k^- = \mathbf{A}\hat{\mathbf{x}}_{k-1}$$

$$\mathbf{P}_k^- = \mathbf{A}\mathbf{P}_{k-1}\mathbf{A}^{\mathrm{T}} + \mathbf{Q}$$

$$\hat{\mathbf{z}}_k = \mathbf{H}\hat{\mathbf{x}}_k^-$$

Correction

Look familiar?

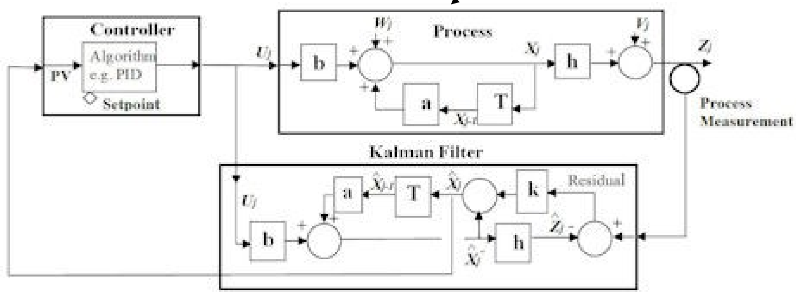
$$\mathbf{K} = \mathbf{P}_k^-\mathbf{H}^{\mathrm{T}}(\mathbf{H}\mathbf{P}_k^-\mathbf{H}^{\mathrm{T}} + \mathbf{R})^{-1}$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K}(\mathbf{z}_k - \hat{\mathbf{z}}_k)$$

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P}_k^-$$

Kalman Filter + Controller

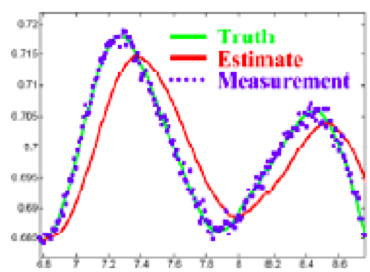
Real system



State estimation

Simulated system
(Kalman Filter)

Problem?



Derivation of the Kalman Filter

System

$$\dot{x}(t) = \mathbf{A}(t)x(t) + \mathbf{B}_w(t)w(t)$$

$$y(t) = \mathbf{H}(t)x(t) + v(t)$$

To derive the Kalman Filter for the system, consider the following:

Cost

$$J = \int_0^{t_f} [w^T(t)\Sigma_w^{-1}w(t) + v^T(t)\Sigma_v^{-1}v(t)]dt + (x(0) - x_0)^T\Sigma_x^{-1}(x(0) - x_0)$$

System

$$\dot{x}(t) = \mathbf{A}(t)\hat{x}(t) + \mathbf{B}_w(t)w(t)$$

$$\tilde{y}(t) = \mathbf{H}(t)x(t) + v(t)$$

Find $\hat{x}(t)$ that minimize J

Derivation of the Kalman Filter

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) - P(t)B_w(t)\Sigma_w B_w^T(t)P(t) + H^T(t)\Sigma_v^{-1}H(t)$$

$$-\dot{z} = (A^T - PB_w\Sigma_w B_w)z - C^T\Sigma_v^{-1}\tilde{y}$$

$$z(0) = 0\ ,\ \ P(0) = \Sigma_x^{-1}$$

$$J(x(t), t) = x^T(t)P(t)x(t) + z^T(t)x(t)$$

$$\hat{x}(t) = \arg\min_x J(x,t) = -P^{-1}(t)z(t)$$

$$X(t) = P^{-1}(t) \quad X(0) = \Sigma_x$$

$$\dot{X}(t) = A(t)X(t) + X(t)A^T(t) + B_w(t)\Sigma_w B_w^T(t) - X(t)H^T(t)\Sigma_v^{-1}H(t)X(t)$$

$$\dot{\hat{x}} = A\hat{x} - XH^T\Sigma_v^{-1}[y - H\hat{x}]$$

Kalman Filter Summary

- What it does

Optimally estimates system position from noisy sensor data

- What it does not do:

Handle nonlinear systems

- However these handle nonlinear systems:

Extended Kalman Filter (EKF)

Particle Filter (PK)

Administrative

- Next week: Saul will cover
  - Nov. 18th: POMDP
  - Nov. 20th: Manipulation & NAMO
- Three lectures left
  - Nov. 25th : Summary
  - Dec. 2nd : Extension of Planning/Control: Language, Hybrid System
  - Dec. 4th : Wrap up
- Due Reminder:
  - Project report: Due Dec. 4th
  - Project report review: Due Dec. 11th
  - Project presentation & presentation evaluation: Dec. 11th