CS 4649/7649
Robot Intelligence: Planning

Linearized Control

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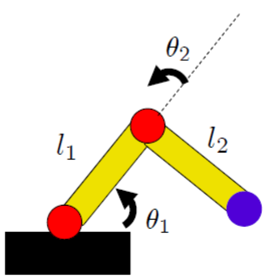
*Slides based in part on Dr. Mike Stilman's slides

Manipulation: Linear Dynamics?



2-DOF Robot Arm

$$\mathbf{T}_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

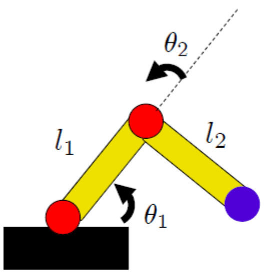
$$\mathbf{T}_2^1 = \begin{bmatrix} c_2 & -s_2 & l_1 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_3^2 = \begin{bmatrix} 1 & 0 & l_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_2^0 = \mathbf{T}_1^0\mathbf{T}_2^1 = \begin{bmatrix} c_1c_2 - s_1s_2 & -c_1s_2 - s_1c_2 & l_1c_1 \\ s_1c_2 + c_1s_2 & -s_1s_2 + c_1c_2 & l_1s_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{12} & -s_{12} & l_1c_1 \\ s_{12} & c_{12} & l_1s_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_3^0 = \mathbf{T}_2^0\mathbf{T}_3^2 = \begin{bmatrix} c_{12} & -s_{12} & l_1c_1 + l_2c_{12} \\ s_{12} & c_{12} & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 \end{bmatrix}$$

Manipulation: Linear Dynamics?



2-DOF Robot Arm

$$\mathbf{T}_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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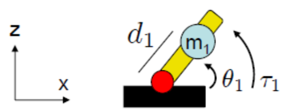
Manipulation: Gravity Compensation



$$x_1 = d_1\cos(\theta_1)$$

Torque due to gravity:

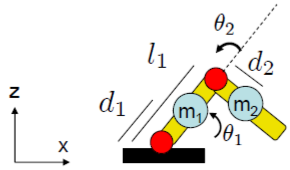
$$\tau_g = -d_1\cos(\theta_1)m_1g$$

$$\tau_1 = -d_1\cos(\theta_1)m_1g + \tau_1^a$$

$$\tau_1^a = -K_p(\theta_1 - \theta_{1d}) - K_d\dot{\theta}_1$$

Better: $\tau_1^a = d_1c_1m_1g - K_p(\theta_1 - \theta_{1d}) - K_d\dot{\theta}_1$

Manipulation: Gravity Compensation



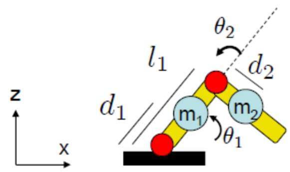
$$x_1 = d_1 c_1$$
$$x_2 = l_1 c_1 + d_2 c_2$$

Torque Due to Gravity:

$$\tau_2^g = -d_2 c_{12} m_2 g$$
$$\tau_1^g = -d_1 c_1 m_1 g - (l_1 c_1 + d_2 c_{12}) m_2 g$$

How do we do Position Control?

Manipulation: Gravity Compensation



$$x_1 = d_1 c_1$$
$$x_2 = l_1 c_1 + d_2 c_2$$

Torque Due to Gravity:

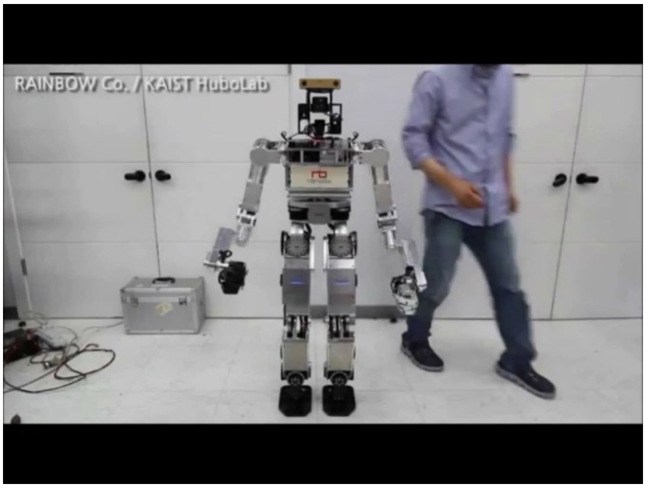
$$\tau_2^g = -d_2 c_{12} m_2 g$$
$$\tau_1^g = -d_1 c_1 m_1 g - (l_1 c_1 + d_2 c_{12}) m_2 g$$

Position Control with Gravity Compensation:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} d_2 c_{12} m_2 \\ d_1 c_1 m_1 + (l_1 c_1 + d_2 c_{12}) m_2 \end{bmatrix} g - \mathbf{K}_p \begin{bmatrix} \theta_1 - \theta_{1d} \\ \theta_2 - \theta_{2d} \end{bmatrix} - \mathbf{K}_d \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Gains are typically diagonal matrices

Manipulation: Gravity & Position



Manipulation: Full Model Control

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \tau$$

Augmented PD:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}_d + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}_d + \mathbf{G}(\mathbf{q}) \boxed{- \mathbf{K}_d(\dot{\mathbf{q}} - \dot{\mathbf{q}}_d) - \mathbf{K}_p(\mathbf{q} - \mathbf{q}_d)} = \tau$$

Computed Torque:

$$\mathbf{M}(\mathbf{q})(\ddot{\mathbf{q}}_d \boxed{- \mathbf{K}_d(\dot{\mathbf{q}} - \dot{\mathbf{q}}_d) - \mathbf{K}_p(\mathbf{q} - \mathbf{q}_d)}) + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \tau$$

Joint Space to Work Space

$\delta \mathbf{x} = \mathbf{J}\delta \mathbf{q}$ $\quad \mathbf{W} = \mathbf{F}^{\mathbf{T}}\mathbf{x}$ $\quad \delta \mathbf{W} = \mathbf{F}^{\mathbf{T}}\delta \mathbf{x}$ $\quad \delta \mathbf{W} = \tau^{\mathbf{T}}\delta \mathbf{q}$

$$\mathbf{F}^{\mathbf{T}}\delta \mathbf{x} = \tau^{\mathbf{T}}\delta \mathbf{q}$$

$$\mathbf{F}^{\mathbf{T}}\mathbf{J}\delta \mathbf{q} = \tau^{\mathbf{T}}\delta \mathbf{q}$$

Open Loop Force Control $\quad \tau = \mathbf{J}^{\mathbf{T}}\mathbf{F}$

Velocity Jacobian for Link Mass i: $\mathbf{J}_{\mathbf{vi}}$ $\quad \mathbf{J}_{\mathbf{vi}}^{\mathbf{T}}\mathbf{g} = ?$

$\mathbf{g} = \text{gravity vector} = \begin{bmatrix} 0 & 0 & 9.8 \end{bmatrix}^T$

Joint Space to Work Space

$\delta \mathbf{x} = \mathbf{J}\delta \mathbf{q}$ $\quad \mathbf{W} = \mathbf{F}^{\mathbf{T}}\mathbf{x}$ $\quad \delta \mathbf{W} = \mathbf{F}^{\mathbf{T}}\delta \mathbf{x}$ $\quad \delta \mathbf{W} = \tau^{\mathbf{T}}\delta \mathbf{q}$

$$\mathbf{F}^{\mathbf{T}}\delta \mathbf{x} = \tau^{\mathbf{T}}\delta \mathbf{q}$$

$$\mathbf{F}^{\mathbf{T}}\mathbf{J}\delta \mathbf{q} = \tau^{\mathbf{T}}\delta \mathbf{q}$$

Open Loop Force Control $\quad \tau = \mathbf{J}^{\mathbf{T}}\mathbf{F}$

Velocity Jacobian for Link Mass i: $\mathbf{J}_{\mathbf{vi}}$ $\quad \mathbf{J}_{\mathbf{vi}}^{\mathbf{T}}\mathbf{g} = ?$

$$\mathbf{G} = -(\mathbf{J}_{v1}^T(m_1\mathbf{g}) + \mathbf{J}_{v2}^T(m_2\mathbf{g})...\mathbf{J}_{vn}^T(m_n\mathbf{g}))$$

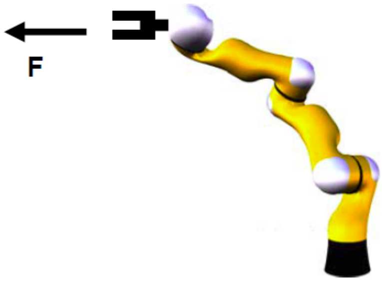
$$\tau = \mathbf{G}(\mathbf{q}) + \mathbf{J}^{\mathbf{T}}\mathbf{F}$$

Manipulation: Workspace Control

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \tau$$

$$\bar{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{x}} + \bar{\mathbf{C}}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{x}} + \bar{\mathbf{G}}(\mathbf{q}) = \mathbf{F}$$

$$\begin{aligned} \bar{\mathrm{M}} &= J^{-T} M J^{T} \\ \bar{\mathrm{C}} &= J^{-T}(C J^{-1} + M \dot{J}^{-1}) \\ \bar{\mathrm{G}} &= J^{-T} G \\ \mathbf{F} &= J^{-T} \tau \end{aligned}$$



Khatib '80, Murray '94

Manipulation: Workspace Control

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \tau$$

$$\bar{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{x}} + \bar{\mathbf{C}}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{x}} + \bar{\mathbf{G}}(\mathbf{q}) = \mathbf{F}$$

$$\begin{aligned} \bar{\mathrm{M}} &= J^{-T} M J^{T} \\ \bar{\mathrm{C}} &= J^{-T}(C J^{-1} + M \dot{J}^{-1}) \\ \bar{\mathrm{G}} &= J^{-T} G \\ \mathbf{F} &= J^{-T} \tau \end{aligned}$$

Computed Torque:

$$\mathbf{F} = \bar{\mathrm{M}}(\mathrm{q})(\ddot{\mathrm{x}}_{\mathrm{d}} + \mathrm{K_d}(\dot{\mathrm{x}} - \dot{\mathrm{x}}_{\mathrm{d}}) - \mathrm{K_p}(\mathrm{x} - \mathrm{x_d})) + \bar{\mathrm{C}}(\mathrm{q},\dot{\mathrm{q}})\dot{\mathrm{x}} + \bar{\mathrm{G}}(\mathrm{q})$$

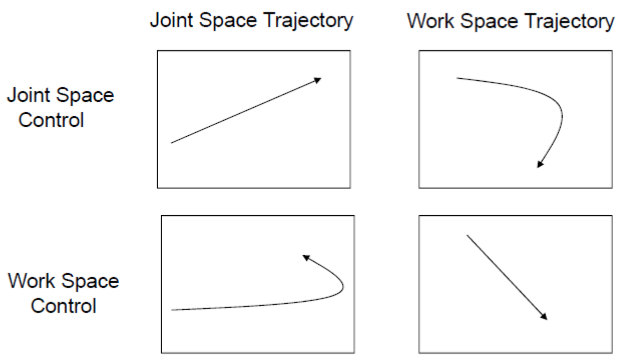
$$\tau = \mathbf{J}^{\mathrm{T}}\mathbf{F}$$

Gravity Augmented PD:

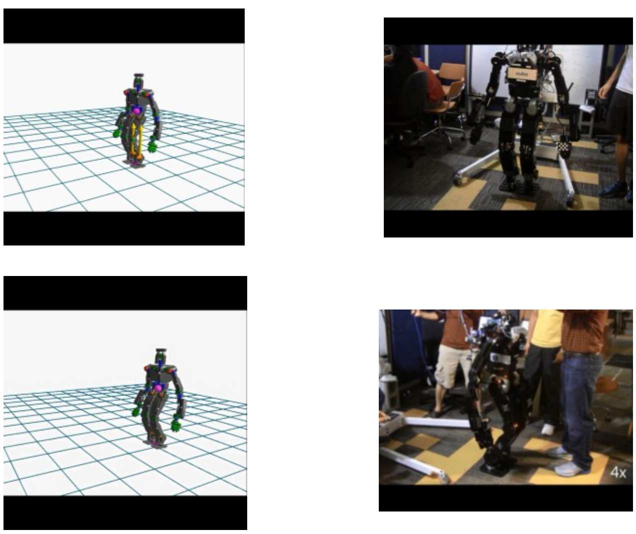
$$\mathbf{F} = \bar{\mathrm{G}}(\mathrm{q}) - \mathrm{K_d}(\dot{\mathrm{x}} - \dot{\mathrm{x}}_{\mathrm{d}}) - \mathrm{K_p}(\mathrm{x} - \mathrm{x_d})$$

$$\tau = \mathbf{J}^{\mathrm{T}}\mathbf{F}$$

Joint Space vs Work Space



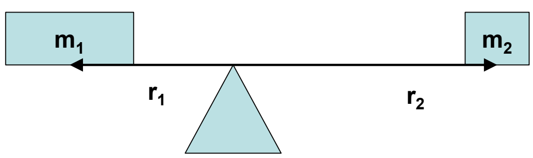
Locomotion: Walking



Balance

Balance

Moment of force (Torque)

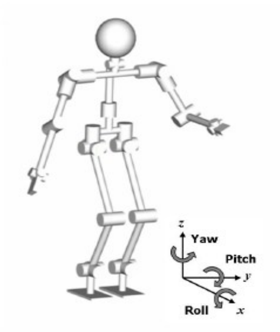


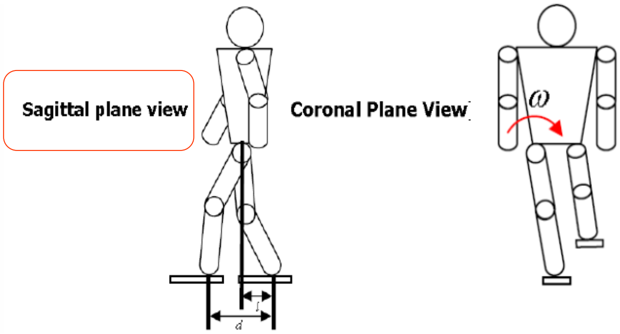
$$\bar{r}_1 \times \bar{F}_1 + \bar{r}_2 \times \bar{F}_2 = 0$$

$$\bar{F}_1 = m_1 \bar{g}$$

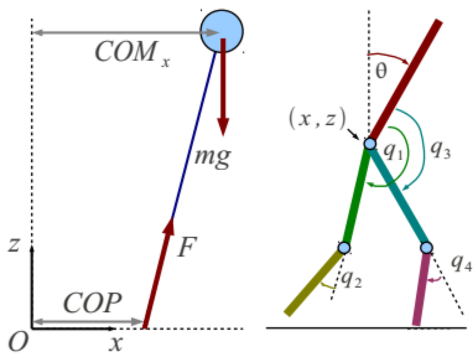
$$\bar{F}_2 = m_2 \bar{g}$$

Biped Walking



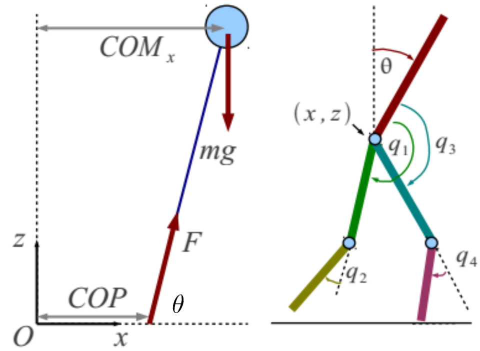


Inverted Pendulum Model



Center of Mass: COM
Center of Pressure: COP

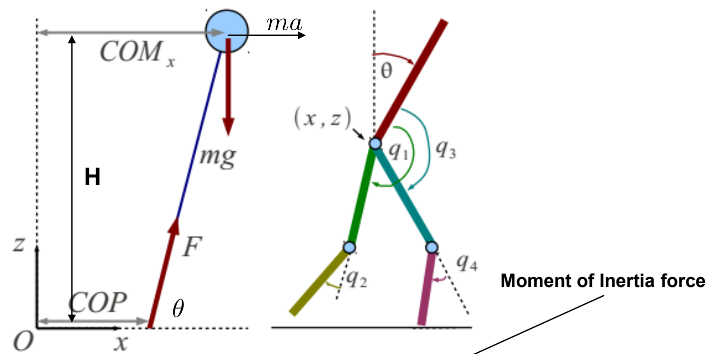
Inverted Pendulum Model



If balanced about O

$$-COM_x mg + COP_x F \sin\theta = 0$$

Inverted Pendulum Model

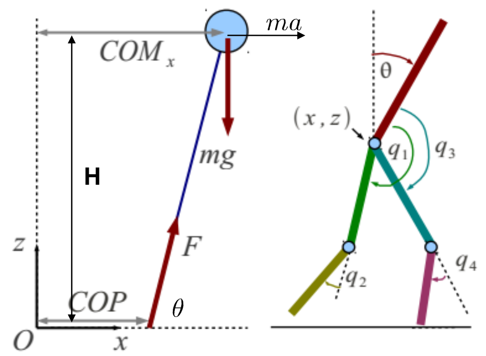


If the mass is moving with some acceleration in x direction

$$-x \cdot m \cdot g + p \cdot F \cdot \sin\theta + H \cdot m \cdot a = 0 \qquad x = COM_x \ , \ p = COP$$

$$p = \frac{x \cdot m \cdot g}{F \cdot \sin\theta} - \frac{H \cdot m \cdot \ddot{x}}{F \cdot \sin\theta} \qquad a = \ddot{x}$$

Inverted Pendulum Model

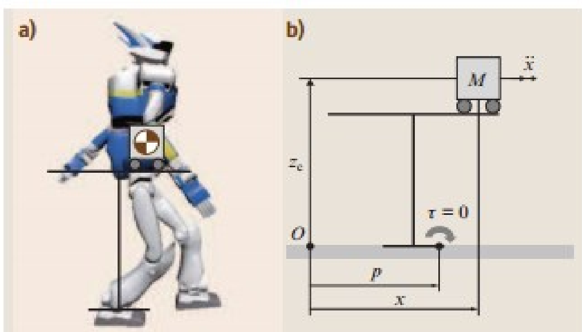


At upright pose, F = mg & ϴ = pi/2

$$-x \cdot m \cdot g + p \cdot F \cdot \sin\theta + H \cdot m \cdot a = 0 \qquad x = COM_x \ , \ p = COP$$

$$p = x - \frac{H}{g}\ddot{x} \qquad a = \ddot{x}$$

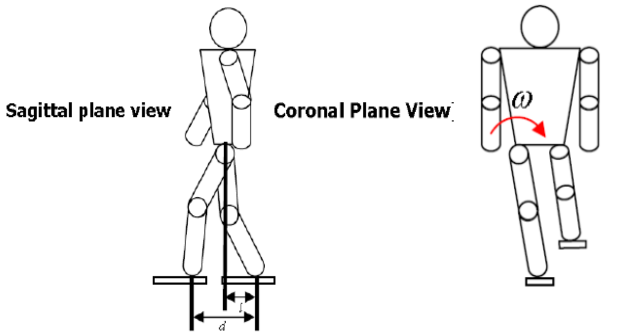
Table-Cart Model



$$p = x - \frac{H}{g}\ddot{x}$$

P = Zero Moment Point (ZMP)

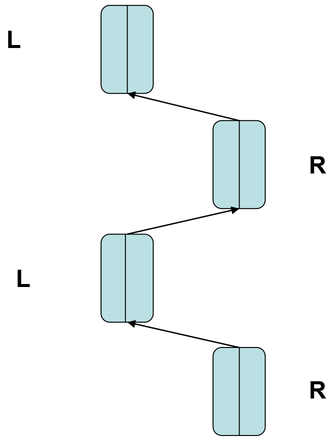
Table-Cart Model



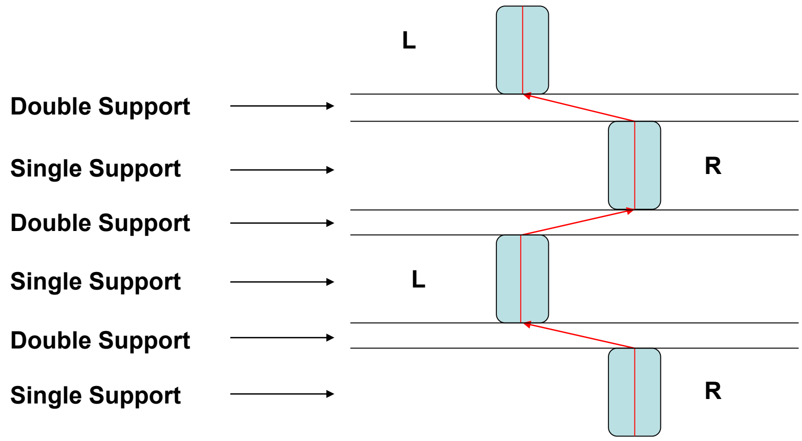
$$p_x = x - \frac{H}{g}\ddot{x}$$

$$p_y = y - \frac{H}{g}\ddot{y}$$

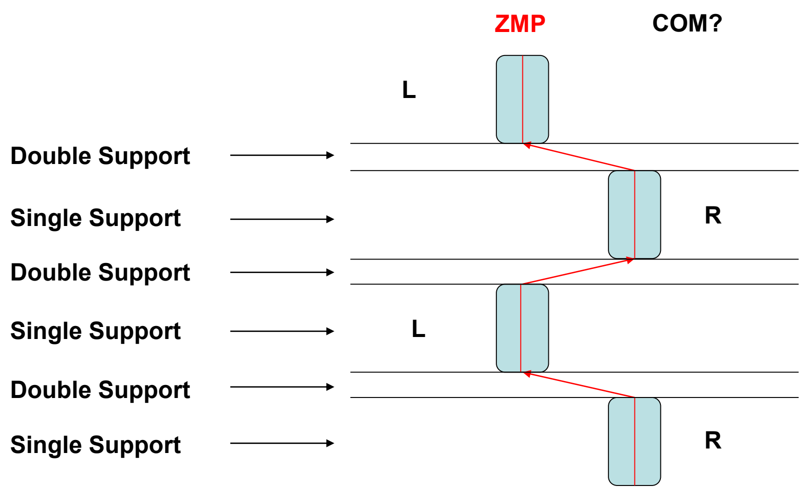
Footsteps: ZMP vs COM



Footsteps: ZMP vs COM



Footsteps: ZMP vs COM



Linear Optimal Control

$$\mathbf{p}_{zmp} = \mathbf{x}_G - T_c^2 \ddot{\mathbf{x}}_G \qquad T_c = \sqrt{\frac{z_G}{g}}$$

$$\mathbf{p}_{zmp} = [x_{zmp}, y_{zmp}]^T \in \mathbb{R}^2$$

$$\mathbf{x}_G = [x_G, y_G]^T \in \mathbb{R}^2 \qquad \frac{d\ddot{\mathbf{x}}_G}{dt} = \mathbf{u} \in \mathbb{R}^2$$

$$\mathbf{X}(t) = [\mathbf{x}^T(t), \dot{\mathbf{x}}^T(t), \ddot{\mathbf{x}}^T(t)]^T \in \mathbb{R}^6$$

$$\mathbf{X}(k+1) = \mathbf{AX}(k) + \mathbf{Bu}(k)$$

$$\mathbf{y}(k) = \mathbf{p}_{zmp} = \mathbf{CX}(k) \in \mathbf{R}^2$$

Linear Optimal Control

$$\mathbf{X}(t) = [\mathbf{x}^T(t), \dot{\mathbf{x}}^T(t), \ddot{\mathbf{x}}^T(t)]^T \in \mathbb{R}^6$$

$$\mathbf{X}(k+1) = \mathbf{AX}(k) + \mathbf{Bu}(k)$$

$$\mathbf{y}(k) = \mathbf{p}_{zmp} = \mathbf{CX}(k) \in \mathbf{R}^2$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_2 & \Delta t \mathbf{I}_2 & \frac{\Delta t^2}{2}\mathbf{I}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & \Delta t \mathbf{I}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix} \in \mathbf{R}^{6\times 6}. \qquad \mathbf{B} = \begin{bmatrix} \frac{\Delta t^3}{6}\mathbf{I}_2 \\ \frac{\Delta t^2}{2}\mathbf{I}_2 \\ \Delta t \mathbf{I}_2 \end{bmatrix} \in \mathbf{R}^{6\times 2}$$

$$\mathbf{C} = [I_2 \ \mathbf{0}_2 \ -T_c^2 I_2] \in \mathbf{R}^{2\times 6}$$

Preview Control

$$\Delta \mathbf{u}(k) = \mathbf{u}(k) - \mathbf{u}(k-1)$$

$$\Delta \mathbf{X}(k) = \mathbf{X}(k) - \mathbf{X}(k-1)$$

$$\Delta \mathbf{y}_d(k) = \mathbf{y}_d(k) - \mathbf{y}_d(k-1)$$

$$\mathbf{y}_d(k+i) = \mathbf{y}_d(k+N) \qquad i > N = \text{preview horizon}$$

$$\mathbf{e}(k) = \mathbf{y}(k) - \mathbf{y}_d(k)$$

$$\Delta \mathbf{X}(k+1) = \mathbf{A}\Delta \mathbf{X}(k) + \mathbf{B}\Delta \mathbf{u}(k)$$

$$\mathbf{e}(k+1) = \mathbf{e}(k) + \mathbf{CA}\Delta \mathbf{X}(k) + \mathbf{CB}\Delta \mathbf{u}(k) - \Delta \mathbf{y}_d(k+1)$$

Preview Control

$$\bar{\mathbf{X}}(\mathrm{k}) = [\mathbf{e}^{\mathrm{T}}(\mathrm{k}), \Delta \mathbf{X}^{\mathrm{T}}(\mathrm{k}), \Delta \mathbf{y}_{\mathrm{d}}^{\mathrm{T}}(\mathrm{k}+1), ..., \Delta \mathbf{y}_{\mathrm{d}}^{\mathrm{T}}(\mathrm{k}+\mathrm{N})]^{\mathrm{T}} \in \mathbb{R}^{\mathrm{p+n+pN}}$$

$$\bar{\mathbf{X}}(k+1) = \bar{\mathbf{A}}\bar{\mathbf{X}}(k) + \bar{\mathbf{B}}\Delta \mathbf{u}(k)$$

$$\bar{\mathbf{A}} = \left[\begin{array}{cc|ccccc} \mathrm{I_p} & \mathrm{CA} & -\mathrm{I_p} & 0 & \dots & \dots & 0 \\ 0 & \mathrm{A} & 0 & 0 & \dots & \dots & 0 \\ \hline 0 & 0 & 0 & \mathrm{I_p} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \mathrm{I_p} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I_p} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \bar{\mathbf{B}} = \left[\begin{array}{c} \mathbf{CB} \\ \mathbf{B} \\ \hline \mathbf{0} \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{0} \end{array}\right]$$

Linear Optimal Control

$$J = \sum_{i=k}^{\infty} [\mathbf{e}^T(i)\mathbf{Q}_e\mathbf{e}(i) + \Delta\mathbf{X}^T(i)\mathbf{Q}_x\Delta\mathbf{X}(i) + \Delta\mathbf{u}^T(i)\mathbf{R}\Delta\mathbf{u}(i)] = \sum_{i=k}^{\infty} \bar{\mathbf{X}}^T(i)\bar{\mathbf{Q}}\bar{\mathbf{X}}(i) + \Delta\mathbf{u}^T(i)\bar{\mathbf{R}}\Delta\mathbf{u}(i)$$

$$\bar{\mathbf{Q}} = \left[\begin{array}{c|c|c} \mathbf{Q}_e & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{Q}_x & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0}_{pN} \end{array}\right]$$

$$\Delta\mathbf{u}^*(k) = -\mathbf{G}_1\mathbf{e}(k) - \mathbf{G}_2\Delta\mathbf{X}(k) - \textstyle\sum_{i=1}^{N} \mathbf{G}_3(i)\Delta\mathbf{y}_d(k+i)$$

$$\mathbf{G}_1 = \mathbf{W}\mathbf{P}\tilde{\mathbf{I}} \qquad \mathbf{G}_2 = \mathbf{W}\mathbf{P}\tilde{\mathbf{F}} \qquad \mathbf{G}_3 = -\mathbf{W}\left([\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{W}\mathbf{P}\tilde{\tilde{\mathbf{A}}}]^T\right)^{k-1}\mathbf{P}\tilde{\mathbf{I}}$$

$$\tilde{\mathbf{B}} = \begin{bmatrix}\mathbf{CB}\\ \mathbf{B}\end{bmatrix} \qquad \tilde{\mathbf{I}} = \begin{bmatrix}\mathbf{I}_p\\ \mathbf{0}_p\end{bmatrix} \qquad \tilde{\mathbf{F}} = \begin{bmatrix}\mathbf{CA}\\ \mathbf{A}\end{bmatrix} \qquad \tilde{\mathbf{Q}} = \begin{bmatrix}\mathbf{Q}_e & \mathbf{0}_{p\times n}\\ \mathbf{0}_{n\times p} & \mathbf{Q}_x\end{bmatrix}$$

$$\tilde{\mathbf{A}} = [\tilde{\mathbf{I}} \quad \tilde{\mathbf{F}}] \qquad \mathbf{W} = [\mathbf{R} + \tilde{\mathbf{B}}^T\mathbf{P}\tilde{\mathbf{B}}]^{-1}\tilde{\mathbf{B}}^T$$

Linear Optimal Control

Discrete Algebraic Riccati Equation (DARE)

$$\mathbf{P} = \tilde{\mathbf{A}}^T\mathbf{P}\tilde{\mathbf{A}} - \tilde{\mathbf{A}}\mathbf{P}\tilde{\mathbf{B}}[\mathbf{R} + \tilde{\mathbf{B}}^T\mathbf{P}\tilde{\mathbf{B}}]^{-1}\tilde{\mathbf{B}}^T\mathbf{P}\tilde{\mathbf{A}} + \tilde{\mathbf{Q}}$$

$$\mathbf{u}^{optimal}(0) = -\mathbf{G}_2\mathbf{X}(0) - \textstyle\sum_{j=1}^{N} \mathbf{G}_3(j)\mathbf{y}_d(j)$$

$$\mathbf{u}^{optimal}(k) = -\mathbf{G}_1\textstyle\sum_{i=0}^{k} \mathbf{e}(i) - \mathbf{G}_2\mathbf{X}(k) - \textstyle\sum_{j=1}^{N} \mathbf{G}_3(j)\mathbf{y}_d(k+j)$$

Error integral **State F/B** **F/F(preview)**

ZMP-Preview Biped Walking